PIVOT TRANSFORMS

ANNEX 1

TANGENT PLANES TO SURFACES

If a surface is defined by two parameters u,v i.e.

$$x=x(u,v)$$
$$y=y(u,v)$$
$$z=z(u,v)$$

and (x,y,z) is the tangent point of a plane while (ξ, η, ζ) are the coordinates of a variable point in that plane then the plane is given by the equation

$$(\xi-x)\frac{\partial(y,z)}{\partial(u,v)}+(\eta-y)\frac{\partial(z,x)}{\partial(u,v)}+(\zeta-z)\frac{\partial(x,y)}{\partial(u,v)}=0$$

where e.g.

$$\frac{\partial(y,z)}{\partial(u,v)}\equiv\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}$$

(c.f. for example *Partial Differentiation* by R.P. Gillespie).

If (x,y,z)=(0,0,0) then

$$\xi\frac{\partial(y,z)}{\partial(u,v)}+\eta\frac{\partial(z,x)}{\partial(u,v)}+\zeta\frac{\partial(x,y)}{\partial(u,v)}=0$$

which gives a plane through the origin parallel to the tangent plane with the determinants as its plane coordinates.

It follows that the direction cosines of the normal to the surface at (x,y,z) are proportional to

$$\left\{\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)},\frac{\partial(x,y)}{\partial(u,v)}\right\}$$

We select the parameters u,v as follows

u = the vertical height of G above O (see main text)
v = the height h of the contour plane.

Then from (13) in the main text we have

$$y(u,v)=\frac{kv(y_3-mx_3)}{1+m^2}$$
$$x(u,v)=-my(u,v)$$
$$z(u,v)=v$$

noting that a+b=e+c=1 (c.f. Figure 9).

Since b=h' and a=1-b, and recalling (11) and (12), we have

$$b=\frac{k_1 v}{k_1 v+k_2(1-v)}$$
$$a=\frac{k_2(1-v)}{k_1 v+k_2(1-v)}$$
$$x_3=\frac{a\left[ex_2+u(x_1-x_2)\right]}{e\left[a-e+u\right]}$$
$$y_3=\frac{a\left[ey_2+u(y_1-y_2)\right]}{e\left[a-e+u\right]}$$

The following partial derivatives are then obtained:

$$\frac{\partial y_3}{\partial u}=\frac{a\left[(a-e)(y_1-y_2)-ey_2\right]}{e(a-e+u)^2}$$
$$\frac{\partial x_3}{\partial u}=\frac{a\left[(a-e)(x_1-x_2)-ex_2\right]}{e(a-e+u)^2}$$
$$\frac{\partial y_3}{\partial v}=-\frac{k_2 b^2}{k_1 ev^2}\frac{(u-e)\left[ey_2+u(y_1-y_2)\right]}{(a-e+u)^2}$$
$$\frac{\partial x_3}{\partial v}=-\frac{k_2 b^2}{k_1 ev^2}\frac{(u-e)\left[ex_2+u(x_1-x_2)\right]}{(a-e+u)^2}$$

Since z=v we have

$$\frac{\partial z}{\partial u}=0$$
$$\frac{\partial z}{\partial v}=1$$

We now require ∂m/∂u and ∂m/∂v which requires us to find a suitable expression for m. Referring to Figure 9 let the equation of the tangent to the circle be y=mx+q. Then since it contains (x_3,y_3) we have $y_3 = mx_3+q$.

It intersects the circle $x^2+y^2-2xx_1-2yy_1+(x_1^2+y_1^2-R^2)=0$ in the points given by

$$(mx+q)^2+x^2-2xx_1-2y_1(mx+q)+(x_1^2+y_1^2-R^2)=0$$

which may be rearranged as a quadratic equation in x:

$$x^2(m^2+1)+2x(mq-y_1m-x_1)+(q^2-2y_1q+x_1^2+y_1^2-R^2)=0$$

Setting the descriminant to zero to give us equal roots (for a tangent) we find

$$m^2(R^2-x_1^2)+2mx_1(y_1-q)+(R^2+2qy_1-y_1^2-q^2)=0$$

Substituting $q=y_3-mx_3$ and simplifying gives

$$m^2[R^2-(x_1-x_3)^2]+2m(x_1-x_3)(y_1-y_3)+R^2-(y_1-y_3)^2=0$$

which is a quadratic equation in m for the two tangents to the circle in Figure 9, in terms of known quantities. Noting that x_1 and y_1 are constant we obtain from this

$$\frac{\partial m}{\partial u}=\frac{m\left[(x_1-x_3)\frac{\partial y_3}{\partial u}+(y_1-y_3)\frac{\partial x_3}{\partial u}\right]-m^2\left[(x_1-x_3)\frac{\partial x_3}{\partial u}+R\frac{\partial R}{\partial u}\right]-(y_1-y_3)\frac{\partial y_3}{\partial u}-R\frac{\partial R}{\partial u}}{m\left[R^2-(x_1-x_3)^2\right]+(x_1-x_3)(y_1-y_3)}$$

Similarly we get

$$\frac{\partial m}{\partial v}=\frac{m\left[(x_1-x_3)\frac{\partial y_3}{\partial v}+(y_1-y_3)\frac{\partial x_3}{\partial v}\right]-m^2\left[(x_1-x_3)\frac{\partial x_3}{\partial v}\right]-(y_1-y_3)\frac{\partial y_3}{\partial v}}{m\left[R^2-(x_1-x_3)^2\right]+(x_1-x_3)(y_1-y_3)}$$

All the subsidiary partial derivatives are given above except ∂R/∂u, which needs to be derived from an expression for R which is independent of m, and determined by the vortex.

In the main text we saw that R is given by

$$R=W\frac{e-u}{u}\left(\frac{u}{e}\right)^{\mu}$$

so clearly

$$\frac{\partial R}{\partial v}=0$$

and differentiating wrt u gives

$$\frac{\partial R}{\partial u}=\frac{W}{u^2}\left(\frac{u}{e}\right)^{\mu}\left[\mu(e-u)-e\right]$$

Now we can find the partial derivatives of x, y and z wrt u and v:

$$
\begin{aligned}
\frac{\partial y}{\partial u} &= \frac{\partial}{\partial u}\left(\frac{kv(y_3 - mx_3)}{1+m^2}\right) \\
&= \frac{kv}{1+m^2}\left\{\frac{\partial y_3}{\partial u} - m\frac{\partial x_3}{\partial u} - x_3\frac{\partial m}{\partial u} - \frac{2\mathrm{m}(y_3 - mx_3)}{1+m^2}\frac{\partial m}{\partial u}\right\} \\
\frac{\partial y}{\partial v} &= \frac{kv}{1+m^2}\left\{\frac{\partial y_3}{\partial v} - m\frac{\partial x_3}{\partial v} - x_3\frac{\partial m}{\partial v} - \frac{2\mathrm{m}(y_3 - mx_3)}{1+m^2}\frac{\partial m}{\partial v}\right\} + \frac{k(y_3 - mx_3)}{1+m^2} \\
\frac{\partial x}{\partial u} &= \frac{\partial(-my)}{\partial u} = -m\frac{\partial y}{\partial u} - y\frac{\partial m}{\partial u} \\
\frac{\partial x}{\partial v} &= -m\frac{\partial y}{\partial v} - y\frac{\partial m}{\partial v}
\end{aligned}
$$

This gives us what we need to calculate

$$\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)} \text{ and } \frac{\partial(x,y)}{\partial(u,v)}$$